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# Shock Waves in Saturated Thermoelastic Porous Media

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**Abstract.** The objective of this paper is to develop and present the macroscopic motion equations for the fluid and the solid matrix, in the case of a saturated porous medium, in the form of coupled, nonlinear wave equations for the fluid and solid velocities. The nonlinearity in the equations enables the generation of shock waves. The complete set of equations required for determining phase velocities in the case of a thermoelastic solid matrix, includes also the energy balance equation for the porous medium as a whole, as well as mass balance equations for the two phase. In the special case of a rigid solid matrix, the wave after an abrupt change in pressure propagates only through the fluid.

**Key words.** Mass, momentum and energy; nonlinear waves; saturated thermoelastic porous media; abrupt pressure change.

## List of Main Symbols

| | |
|---|---|
| $C_f$ | Specific heat of fluid at constant volume. |
| $C_s$ | Specific heat of solid at constant strain. |
| $C_r$ | Shape factor. |
| $E$ | Young's modulus of elasticity. |
| $f$ | As subscript, symbol denoting fluid. |
| $f^H_{\alpha\to\beta}$ | Rate of transfer of heat from the $\alpha$-phase to the $\beta$ one, across their common microscopic interface, per unit volume of porous medium. |
| $\mathbf{F}$ | Body force. |
| $g$ | Gravity acceleration. |
| $H$ | As superscript, symbol denoting heat. |
| $\mathbf{I}$ | Unit tensor. |
| $\mathbf{J}^H_\alpha$ | Macroscopic conductive heat flux of $\alpha$-phase. |
| $\mathbf{J}^{*H}_\alpha$ | Dispersive heat flux of $\alpha$-phase. |
| $p$ | Pressure. |
| $\mathbf{q}_\alpha$ | Specific discharge of $\alpha$-phase. |
| $\mathbf{q}_r$ | Specific discharge of $\alpha$-phase, relative to the solid. |
| $s$ | As subscript, symbol denoting solid. |
| $t$ | Time. |
| $T_\alpha$ | Temperature of $\alpha$-phase. |
| $\mathbf{T}^*$ | Tortuosity. |
| $U_0$ | Volume of domain of REV ($\mathcal{U}_0$). |
| $\mathbf{V}_\alpha$ | Mass weighted velocity of $\alpha$-phase. |
| $\mathbf{w}$ | Displacement of solid skeleton. |
| $\mathbf{w}_s$ | Displacement of solid. |
| $\mathbf{x}$ | Position vector. |
| $z$ | Vertical coordinate (positive upward). |

*Greek Letters*

| | |
|---|---|
| $\alpha$ | As subscript, symbol for an $\alpha$-phase. |
| $\alpha^{*E}$ | Transfer coefficient of $E$. |
| $\beta$ | As subscript, symbol for a $\beta$-phase. A symbol for all other phases, except $\alpha$. |
| $\beta_p$ | Coefficient of fluid compressibility at constant pressure. |
| $\beta_T$ | Coefficient of fluid compressibility at co |

| | |
|---|---|
| $\mathscr{S}_{\alpha\beta}$ | Area of surface of contact of $\alpha$-phase with all other phases (denoted by $\beta$) within the REV. |
| $\delta_{ij}$ | Components of Kronecker delta. |
| $\Delta_\alpha$ | Hydraulic radius of $\alpha$-phase. |
| $\varepsilon$ | Volumetric strain. First invariant of $\boldsymbol{\varepsilon}$. |
| $\boldsymbol{\varepsilon}$ | Strain tensor. |
| $\dot{\boldsymbol{\varepsilon}}$ | Rate of strain. |
| $\varepsilon_{sk}$ | Dilatation of the solid matrix. |
| $\varepsilon_s$ | Dilatation of the solid phase. |
| $\eta$ | Coefficient of thermoelasticity. |
| $\lambda_\alpha$ | Thermal conductivity of $\alpha$-phase. |
| $\lambda''_s, \mu'_s$ | Lamé's constants of an elastic solid. |
| $\mu_\alpha$ | Dynamic viscosity of a fluid $\alpha$-phase. |
| $\nu$ | Poisson's ratio. |
| $\mathbf{v}_\alpha$ | Outward unit vector $\mathscr{S}_{\alpha\beta}$. |
| $\rho_\alpha$ | Mass density of $\alpha$-phase. |
| $\rho_b$ | Bulk density of porous medium. |
| $\boldsymbol{\varepsilon}_\alpha$ | Stress tensor in an $\alpha$-phase. |
| $\boldsymbol{\sigma}'_s$ | Effective stress. |
| $\tau$ | Shear stress. |
| $\phi$ | Porosity. |

*Special Symbols*

| | |
|---|---|
| $\overline{(\ldots)}$ | Average, volume average, or phase average of $(\ldots)(=1/U_0 \int_{\mathscr{U}_0}(\ldots)\,dU)$. |
| $\overline{(\ldots)}^\alpha$ | Intrinsic phase average of $(\ldots)$ $(=1/U_{o\alpha}\int_{u_o}(\ldots)\,dU)$. |
| $\widetilde{(\ldots)}^{\alpha\beta}$ | Average of $(..)$ over $\mathscr{S}_{\alpha\beta}$-surface. |
| $\dfrac{D_{E(\ldots)}}{Dt}$ | material derivative of $(\ldots)$, as observed |

## 1. Introduction

Inertial effects appear in the basic macroscopic momentum balance equation (=*motion equations*), for a fluid phase that occupies the entire void space. Our objective in this paper is to develop the motion equations for the fluid and the solid as *wave equations*, written in terms of phase velocities.

Biot's (1956) work on compressional wave propagation was probably the first one that employed the fundamentals of porous medium mechanics. A large number of papers have appeared in the literature following Biot's pioneering work. A rather extensive literature survey was presented by Corapcioglu (1991).

The complete model, at the macroscopic level, includes mass, momentum and energy balance equations for the fluid and for the solid skeleton, as well as constitutive relations that describe the behavior of the particular materials involved.

## 2. The Conceptual Model

The conceptual model may be summarized by the following set of simplifying assumptions:

[A.1] The fluid is Newtonian and compressible.

[A.2] The solid phase preserves its volume.

[A.3] The solid matrix is thermoelastic, but the solid matrix is assumed to undergo only small deformations.

[A.4] The dispersive flux of momentum is much smaller than its advective one and may, therefore, be neglected.

[A.5] The dispersive and diffusive mass fluxes of a fluid, and the dispersive flux of the solid, are much smaller than the corresponding advective ones, and may, therefore, be neglected.

[A.6] The microscopic solid-fluid interfaces are material surfaces with respect to the mass of both phases.

[A.7] Energy processes for the fluid and for the solid are assumed reversible, and specific heat at constant volume, $C_f$ for the fluid and specific heat at constant strain, $C_s$ for the solid, are assumed constant.

[A.8] There exist no external energy sources.

[A.9] The energy associated with viscous dissipation is assumed negligible, i.e. $|\boldsymbol{\tau} : \nabla \mathbf{V}_f| \ll |p\, \delta \nabla \cdot \mathbf{V}_s|$.

[A.10] The stress-strain relationship for the solid, at the microscopic level, and for the solid matrix, at the macroscopic one, have the same form.

## 3. Momentum Balance Equations

Making use of assumptions [A.4] and [A.6], the macroscopic momentum balance equations for a fluid phase, $f$, and for a solid one, $s$, are

$$\frac{\partial\, \phi \rho_f \mathbf{V}_f}{\partial t} = -\nabla \cdot \phi(\rho_f \mathbf{V}_f \mathbf{V}_f - \boldsymbol{\sigma}_f) + \phi \rho_f \mathbf{F} + \frac{1}{U_0} \int_{\mathscr{S}_{fs}} \hat{\boldsymbol{\sigma}}_f \cdot \mathbf{v}_f \, \mathrm{d}S, \tag{1}$$

and

$$\frac{\partial (1-\phi)\rho_s \mathbf{V}_s}{\partial t} = -\nabla \cdot (1-\phi)(\rho_s \mathbf{V}_s \mathbf{V}_s - \boldsymbol{\sigma}_s) + (1-\phi)\rho_s \mathbf{F} + \frac{1}{U_0} \int_{\mathscr{S}_{sf}} \hat{\boldsymbol{\sigma}}_s \cdot \mathbf{v}_s \, \mathrm{d}S, \tag{2}$$

respectively, where

$$\rho_b = \phi \rho_f + (1-\phi)\rho_s \tag{3}$$

denotes the bulk density of the porous medium,

$$\mathbf{V}_r = \mathbf{V}_f - \mathbf{V}_s \tag{4}$$

denotes the velocity of the fluid relative to the solid, and

$\rho_\alpha$ = Intrinsic phase average mass density of an $\alpha$-phase ($\alpha = f, s$).
$\mathbf{V}_\alpha$ = Intrinsic phase average mass weighted velocity of an $\alpha$-phase.
$\phi$ = Porosity.
$\hat{\boldsymbol{\sigma}}_\alpha$ = Stress tensor of an $\alpha$-phase, at the microscopic level.
$\boldsymbol{\sigma}_\alpha$ = Intrinsic phase average stress tensor of an $\alpha$-phase.
$\mathscr{S}_{\alpha\beta}$ = Microscopic surface separating the $\alpha$-phase from all other ($\beta-$) phase(s) within an REV.
$\mathbf{v}_\alpha$ = Outward unit vector on $\mathscr{S}_{\alpha\beta}$.
$\mathbf{F}$ = Body force due to gravity ($= -g\nabla z$), per unit mass.
$g$ = Gravity acceleration.
$z$ = Altitude.

In these equations, the surface integrals express the exchange of momentum between the two phases. This exchange is eliminated by summing up the two equations. We obtain the momentum balance equation for the porous medium as a whole, in the form

$$\frac{\partial}{\partial t}(\phi\rho_f\mathbf{V}_r + \rho_b\mathbf{V}_s)$$
$$= -\nabla\cdot[\phi\rho_f(\mathbf{V}_r\mathbf{V}_r + \mathbf{V}_r\mathbf{V}_s + \mathbf{V}_s\mathbf{V}_r) + \rho_b\mathbf{V}_s\mathbf{V}_s] +$$
$$+ \nabla\cdot\boldsymbol{\sigma}_t - \rho_b g\nabla z, \tag{5}$$

in which

$$\boldsymbol{\sigma}_t \equiv \phi\boldsymbol{\sigma}_f + (1-\phi)\boldsymbol{\sigma}_s \tag{6}$$

is the total (macroscopic) stress within the porous medium domain.

For a fluid phase

$$\boldsymbol{\sigma}_f = \boldsymbol{\tau}_f - p\mathbf{I}, \tag{7}$$

where $\boldsymbol{\tau}_f$, and $p$ denote the fluid's shear stress and pressure, respectively, and $\mathbf{I}$ is the unit tensor.

Introducing Terzaghi's (1925) concept of *effective stress*, $\boldsymbol{\sigma}'_s$ (see also Bear and Bachmat, 1990), which is the stress that produces strain in the solid skeleton, we may express the total stress tensor in the form

$$\boldsymbol{\sigma}_t = \boldsymbol{\sigma}'_s + \boldsymbol{\sigma}_f, \tag{8}$$

with the effective stress defined by

$$\boldsymbol{\sigma}'_s = (1-\phi)(\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_f). \tag{9}$$

Here, $\boldsymbol{\sigma}_s$ and $\boldsymbol{\sigma}_f$ are intrinsic phase averages of the corresponding microscopic stresses, while $\boldsymbol{\sigma}_t$ and $\boldsymbol{\sigma}'_s$ are phase averages.

The macroscopic momentum balance equation for the fluid alone, (1), written in indicial notation, can be developed to the form (Bear and Bachmat, 1990)

$$\frac{\partial}{\partial t}\phi\rho_f(V_{ri} + V_{si})$$
$$= -\frac{\partial}{\partial x_j}[\phi\rho_f(V_{ri}V_{rj} + V_{ri}V_{sj} + V_{rj}V_{si} + V_{si}V_{sj})] -$$
$$-\phi\left(\frac{\partial p}{\partial x_j} + \rho_f g\frac{\partial z}{\partial x_j}\right)T^*_{ji} + \mu_f\left[\frac{\partial^2\phi V_{ri}}{\partial x_j\,\partial x_j} + \frac{\partial}{\partial x_j}\left(\phi\frac{\partial V_{si}}{\partial x_j}\right)\right] +$$
$$+ (\mu_f + \lambda''_f)\left\{\frac{\partial^2\phi V_{rj}}{\partial x_i\,\partial x_j} - \frac{1}{\phi}\left[\frac{\partial\phi}{\partial t} + \frac{\partial\phi(V_{rj} + V_{sj})}{\partial x_j}\right]\frac{\partial\phi}{\partial x_i} +\right.$$
$$\left. + \frac{\partial}{\partial x_i}\left(\phi\frac{\partial V_{sj}}{\partial x_j}\right)\right\} - \mu_f\alpha_{ij}\frac{C}{\Delta_f^2}\phi V_{rj}, \tag{10}$$

where $T^*_{ij}, \alpha_{ij}$ = Tensorial coefficients that constitute macroscopic representations of the microscopic configuration of the $\mathscr{S}_{fs}$-interface, within the REV.

$\mu_f =$ Fluid's (intrinsic phase average) viscosity.
$\lambda_f'' + \frac{2}{3}\mu_f =$ Fluid's (intrinsic phase average) second viscosity.
$C_v =$ Macroscopic dimensionless shape factor.
$\Delta_f =$ Hydraulic radius of the fluid occupied void space.

The coefficient $T^*_{ji}$, transforms the local body force into a macroscopic one. The coefficient $\alpha_{ij}$, introduces the effect of the configuration of the solid-fluid surface in the term that transforms part of the force resisting the flow at a point to an averaged resistance force at the fluid-solid interface.

The last term on the l.h.s. of (10) expresses the drag, due to momentum transfer at the fluid-solid interface, i.e.

$$\frac{1}{U_0}\int_{\mathscr{S}_{fs}} \tau_{ij} v_j \, dS = -\mu \frac{C_v}{\Delta_f^2} \alpha_{ij} q_{rj}.$$

Thus, Equations (5) and (10) constitute two coupled motion equations for the velocities $\mathbf{V}_f$ and $\mathbf{V}_s$, or for $\mathbf{V}_r$ and $\mathbf{V}_s$. However, since these equations also contain densities and stresses, additional equations are required in order to obtain a complete set of equations.

## 4. Mass Balance Equations

### 4.1. FOR THE FLUID

With assumption [A.2] and [A.5], the macroscopic mass balance equation for the fluid is

$$\frac{\partial \phi \rho_f}{\partial t} = -\nabla \cdot (\phi \rho_f \mathbf{V}_f). \tag{11}$$

### 4.2. FOR THE SOLID

With assumption [A.4] and [A.5], the macroscopic mass balance equation for the solid is

$$\frac{\partial (1-\phi)\rho_s}{\partial t} = -\nabla \cdot \{(1-\phi)\rho_s \mathbf{V}_s\}, \tag{12}$$

or, since we have assumed small solid deformations, implying $\rho_s =$ const. in the equivalent form

$$\nabla \cdot \mathbf{V}_s = -\frac{1}{1-\phi}\frac{D_s(1-\phi)}{Dt} = \frac{D_s \varepsilon_{sk}}{Dt}, \tag{13}$$

where

$$\frac{D_\alpha(\ldots)}{Dt} \equiv \frac{\partial(\ldots)}{\partial t} + \mathbf{V}_\alpha \nabla(\ldots).$$

denotes the *material derivative* of (. . .), and $\varepsilon_{sk}$ denotes the (macroscopic) dilatation of *the solid skeleton*, i.e., of the porous medium as a whole. It is the first invariant of the skeleton's strain tensor, $\boldsymbol{\varepsilon}_{sk}$.

The skeleton's strain can be expressed in terms of the (macroscopic, i.e., intrinsic phase average) displacement vector, **w**, in the form

$$\varepsilon_{skij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right), \tag{14}$$

with the volumetric strain (=dilatation), $\varepsilon_{sk}$, expressed by

$$\varepsilon_{sk} = \varepsilon_{skii} = \nabla \cdot \mathbf{w}. \tag{15}$$

In this equation, $\boldsymbol{\varepsilon}_{sk}$, denotes the intrinsic phase average strain of the solid. In using (14), we have assumed that form of the *compatibility equation* of the solid and of the solid matrix are the same.

### 4.3. FOR THE POROUS MEDIUM AS A WHOLE

By summing up (11) and (12), we obtain the mass balance equation for the porous medium as a whole, in the form

$$\frac{\partial \rho_b}{\partial t} = -\nabla \cdot (\rho_b \mathbf{V}_s + \phi \rho_f \mathbf{V}_r). \tag{16}$$

## 5. Energy Balance Equations

With assumptions [A.7] through [A.9], Bear and Bachmat (1990) present the following macroscopic energy balance equations. For the fluid phase

$$\begin{aligned}\frac{\partial}{\partial t}(\phi \rho_f C_f T_f)\\ &= -\nabla \cdot \phi\{\rho_f C_f T_f(\mathbf{V}_r + \mathbf{V}_s) + \mathbf{J}_f^H + \mathbf{J}_f^{*H}\} - \\ &\quad - f_{f\to s}^H - T_f \frac{\beta_T}{\beta_p}\bigg|_{\rho_f} \nabla \cdot \phi(\mathbf{V}_r + \mathbf{V}_s),\end{aligned} \tag{17}$$

where $\mathbf{J}_\alpha^H$ = Conductive heat flux in the $\alpha$-phase, at the macroscopic level.
$\mathbf{J}_\alpha^{*H}$ = Dispersive heat flux in the $\alpha$-phase.
$f_{\alpha\to\beta}^H$ = Rate of heat transferred from the $\alpha$-phase to the $\beta$-one, across their common interface within the REV, per unit volume of the latter.
$\beta_p$ = Coefficients of fluid compressibility at constant temperature.
$\beta_T$ = Coefficient of fluid thermal (volumetric) expansion at constant pressure, with

$$\beta_p = \frac{1}{\rho_f}\frac{\partial \rho_f}{\partial p_f}\bigg|_{T_f} \quad \text{and} \quad \beta_T = -\frac{1}{\rho_f}\frac{\partial \rho_f}{\partial T_f}\bigg|_{p_f}. \tag{18}$$

Since we consider here the general case of $T_f \neq T_s$, we may express the rate of heat transfer, $f^H_{s\to f}$, by

$$f^H_{s\to f} = \alpha^{*H}(T_s - T_f), \tag{19}$$

where $\alpha^{*H}$ is a heat transfer coefficient.

For a *thermo-elastic* isotropic solid phase, the macroscopic heat balance equation is given by

$$\begin{aligned}\frac{\partial}{\partial t}(1-\phi)\rho_s C_s T_s \\ &= -\nabla\cdot(1-\phi)\{\rho_s C_s T_s \mathbf{V}_s + \mathbf{J}^H_s\} - \\ &\quad - f^H_{s\to f} - T_s \bar{\eta}\frac{\mathrm{D}_s \varepsilon_{sk}}{\mathrm{D}t},\end{aligned} \tag{20}$$

where $\bar{\eta}(=(1-\phi)\bar{\eta}^s)$ is a macroscopic coefficient that appears in the constitutive equation for the solid matrix. In writing the last term on the r.h.s. of (20), we have assumed that $\nabla\cdot\mathbf{V}_S = 0$, or $\rho_s = \text{const.}$, at the microscopic level (i.e., isochoric motion).

For an isotropic thermoelastic solid

$$\eta\delta_{ij} = -\left.\frac{\partial\sigma_{sij}}{\partial T_s}\right|_{\varepsilon_{sk} = \text{const.}}.$$

We note that in (17) and (20), we have

$$f^H_{s\to f} = -f^H_{f\to s}.$$

In order to express these balance equations in terms of average temperatures as the only state variables, we have to introduce appropriate expressions for the macroscopic conductive fluxes, $\mathbf{J}^H_f$ and $\mathbf{J}^H_s$, as well as for the terms that express the (average) rate of exchange of heat between the phases (e.g., Bear and Bachmat, 1990).

## 6. Constitutive Relations

These express stress-strain relations for particular (fluid and solid) materials, and diffusive fluxes of particular extensive quantities. Obviously, we are interested here in the macroscopic form of these relations. These can be obtained by averaging their microscopic counterparts over an REV.

However, subject to certain simplifying assumptions, the averaged relations take on forms that are similar to their microscopic counterparts, except that the coefficients that appear in them have values (to be determined experimentally) that are different from the corresponding microscopic ones.

6.1. STRESS-STRAIN RELATIONSHIP FOR AN ISOTROPIC THERMOELASTIC SOLID MATRIX THAT UNDERGOES SMALL DEFORMATIONS

With assumption [A.10], this relationship, takes the form

$$\sigma'_{sij} = \lambda''_s \frac{\partial w_{sl}}{\partial x_l} \delta_{ij} + \mu'_s \left( \frac{\partial w_{si}}{\partial x_j} + \frac{\partial w_{sj}}{\partial x_i} \right) - \eta (T_s - T_{s0}) \delta_{ij}, \tag{21}$$

where $\lambda''_s$ and $\mu'_s$ are macroscopic Lamé constants, $\eta$ is the coefficient of thermal stress of the solid matrix, and $\delta$ denotes Kronecker's delta.

With the same assumption, at the macroscopic level, the constitutive relationship for a solid matrix, has the general form

$$\sigma'_{sij} = \sigma'_{sij}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}, T).$$

6.2. STRESS-RATE OF STRAIN RELATIONSHIP FOR A NEWTONIAN COMPRESSIBLE FLUID

At the microscopic level, this constitutive relationship takes the form

$$\tau_{fij} = \mu_f \left( \frac{\partial V_{fi}}{\partial x_j} + \frac{\partial V_{fj}}{\partial x_i} \right) + \lambda''_f \frac{\partial V_{fk}}{\partial x_k} \delta_{ij}. \tag{22}$$

By averaging this equation, assuming

$$V_i|_{\mathscr{S}_{fs}} = V_{si}|_{\mathscr{S}_{fs}} = \tilde{V}^{fs}_{si} = \bar{V}^{s}_{si}, \qquad \tilde{V}^{fs}_{si} = \frac{1}{S_{fs}} \int_{\mathscr{S}_{sf}} V_{si}\, \mathrm{d}S,$$

we obtain

$$\begin{aligned} \overline{\tau_{fij}} &= \overline{\mu'_f} \left( \frac{\partial \phi V_{ri}}{\partial x_j} + \frac{\partial \phi V_{rj}}{\partial x_i} \right) - \overline{\lambda''^{f}_f} \frac{\partial \phi V_{rk}}{\partial x_k} \delta_{ij} + \\ &\quad + 2\phi \overline{\mu'_f} \frac{\mathrm{D}_s \varepsilon_{skij}}{\mathrm{D}t} + \phi \overline{\lambda''^{f}_f} \frac{\mathrm{D}_s \varepsilon_{skij}}{\mathrm{D}t}. \end{aligned} \tag{23}$$

6.3. FLUID COMPRESSIBILITY

$$\rho_f = \rho_f(p, T_f). \tag{24}$$

## 7. Summary of Equations and Variables

At this stage, we have stated all the equations, balance equations, constitutive relations and definitions, included in our model. Altogether, the state variables and the equation are listed in Table I.

Table I. Summary of complete model for saturated flow of a compressible Newtonian fluid in an isotropic linearly thermoelastic porous medium

| Equations | Dependent variables |
|---|---|
| Solid's mass balance equation, (12) | $\mathbf{V}_s, \phi$ |
| Mass balance for porous medium, (16) | $\rho_f, \rho_b, \mathbf{V}_r$ |
| Momentum balance for porous medium, (5) | $\boldsymbol{\sigma}_t$ |
| Fluid's momentum balance, (10) | $p$ |
| Fluid's energy balance, (17) | $T_f$ |
| Solid's energy balance, (20) | $T_s \boldsymbol{\varepsilon}_{sk}$ |
| Solid matrix constitutive relation (21) | $\boldsymbol{\sigma}'_s, \mathbf{w}$ |
| Fluid's constitutive relation, (23) | $\boldsymbol{\tau}$ |
| Fluid's equation of state, (24) | |
| Skeleton's compatability equation, (14) | |
| Volumetric strain (dilatation), (15) | |
| Definition of effective stress (8) | $\boldsymbol{\sigma}_f$ |
| Density of porous medium, (3) | |
| Fluid's stress (7) | |
| Total: 14 equations | 14 variables |

## 8. Wave Equations

To obtain some feeling for the nature of the changes in $\mathbf{V}_f(\mathbf{x}, t)$ and $\mathbf{V}_s(\mathbf{x}, t)$, let us introduce some additional simplifying assumptions.

- Isothermal conditions
- $\mathbf{V}_\alpha \equiv \dfrac{\mathrm{D}_s \mathbf{w}_\alpha}{\mathrm{D}t} \approx \dfrac{\partial \mathbf{w}_\alpha}{\partial t}$
- The viscous drag within the fluid is much smaller than that at the solid-fluid interface, i.e.,

$$\left|\frac{\partial \phi \overline{\tau'_{ij}}}{\partial x_j}\right| \ll \left|\frac{1}{U_0}\int_{\mathscr{S}_{sf}} \tau_{ij} v_{fj}\, \mathrm{d}S\right|.$$

Under these assumptions, the momentum balance for the fluid, when accounting for the mass balance equation, reduces to form

$$\rho_f \frac{\partial \mathbf{V}_f}{\partial t} + \rho_f \mathbf{V}_f \cdot \nabla \mathbf{V}_f$$
$$= -(\nabla p + \rho_f g \nabla z)\mathbf{T}^*_f - \mu_f \boldsymbol{\alpha} \frac{C_\tau}{\Delta_f^2} \frac{\partial \mathbf{V}_r}{\partial t}. \tag{25}$$

For the solid phase, by subtracting this equation from that written for the momentum balance equation written for the porous medium as a whole, taking into account the mass balance equations, we obtain

$$(1-\phi)\rho_s \frac{\partial \mathbf{V}_s}{\partial t} + (1-\phi)\rho_s \mathbf{V}_s \cdot \nabla \mathbf{V}_s$$
$$= \nabla \cdot \boldsymbol{\sigma}_t - \rho_b g \nabla z + \phi(\nabla p + \rho_f g \nabla z)\mathbf{T}_f^* +$$
$$+ \mu_f \boldsymbol{\alpha} \cdot \frac{C_v}{\Delta_f^2} \frac{\partial \mathbf{V}_r}{\partial t}. \tag{26}$$

Obviously, a complete model will still include appropriate constitutive equations and equations of state.

Let us further simplify the conceptual model presented above, by assuming that

- the solid matrix is rigid and stationary.

For such matrix, Bear and Sorek (1990) developed and presented the various forms of the fluid's mass and momentum balance equations that evolve (in time) following an abrupt change in pressure at the domain's boundary. They show that

*First time stage* Immediately following an abrupt pressure change, the pressure impulse propagates without any attenuation.

*Second time stage* Following this initial (very short) period, there follows a period in which, while the fluid is practically at rest, the pressure propagation is governed by a momentum balance equation that has the form

$$\rho_f \frac{\partial \mathbf{V}_f}{\partial t} + \rho_f \mathbf{V}_f \cdot \nabla \mathbf{V}_f = -(\nabla p + \rho_f g \nabla z) \cdot \mathbf{T}_f^*. \tag{27}$$

We note that (25) through (27) are *wave equations*. To verify that these are indeed different forms of the wave equations, let us rewrite these equations in the general form

$$\frac{\partial \mathbf{V}_\alpha}{\partial t} + \mathbf{V}_\alpha \cdot \nabla \mathbf{V}_\alpha = \mathbf{F}_\alpha, \tag{28}$$

where $\mathbf{F}_\alpha$ denotes a driving force associated with the motion of the $\alpha$-phase.

If we now differentiate (28) with respect to time, we obtain

$$\frac{\partial^2 \mathbf{V}_\alpha}{\partial t^2} + \frac{\partial \mathbf{V}_\alpha}{\partial t} \cdot \nabla \mathbf{V}_\alpha + \mathbf{V}_\alpha \cdot \nabla \left( \frac{\partial \mathbf{V}_\alpha}{\partial t} \right) = \frac{\partial \mathbf{F}_\alpha}{\partial t}. \tag{29}$$

In view of (28), when substituting the expression for $\partial \mathbf{V}_\alpha / \partial t$ into (29), we obtain

$$\frac{\partial^2 \mathbf{V}_\alpha}{\partial t^2} - (\mathbf{V}_\alpha \mathbf{V}_\alpha \nabla \nabla) \mathbf{V}_\alpha$$
$$= \frac{\partial \mathbf{F}_\alpha}{\partial t} - \nabla(\mathbf{V}_\alpha \cdot \mathbf{F}_\alpha) + \mathbf{F}_\alpha \times (\nabla \times \mathbf{V}_\alpha) +$$
$$+ \mathbf{V}_\alpha \times (\nabla \times \mathbf{F}_\alpha) + 2\left[ \left( \nabla \frac{V_\alpha^2}{2} - \mathbf{V}_\alpha \times \nabla \times \mathbf{V}_\alpha \right) \cdot \nabla \right] \mathbf{V}_\alpha. \tag{30}$$

Whenever, as an approximation, we neglect the *vorticity*, i.e., we assume $\nabla \times \mathbf{V}_\alpha = 0$, we obtain

$$\frac{\partial^2 \mathbf{V}_\alpha}{\partial t^2} - (\nabla \cdot \mathbf{V}_\alpha)\nabla \frac{V_\alpha^2}{2} = \frac{\partial \mathbf{F}_\alpha}{\partial t} - \nabla(\mathbf{F}_\alpha \cdot \mathbf{V}_\alpha) + \mathbf{V}_\alpha \nabla \cdot \nabla \frac{V_\alpha^2}{2}. \tag{31}$$

We note that the last two equations represent nonlinear wave equations. We have thus demonstrated that (25) through (27) may describe shock waves.

## 9. Conclusions

We have presented a complete set of macroscopic equations that describe fluid pressure changes and fluid and solid temperatures and displacements, in a saturated thermoelastic porous medium domain. The set includes mass, momentum and energy balance equations, constitutive relations and definitions. Included in the balance equations are inertial effects, as well as drag within the fluid and energy sources due to fluid compressibility and solid matrix deformability.

Some simple examples were presented in order to show that, under certain conditions, the solid and fluid velocities obey coupled wave equations.

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